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ASYMPTOTICS OF STRESSES AND STRAIN RATES NEAR THE TIP OF A TRANSVERSE SHEAR CRACK IN A MATERIAL WHOSE BEHAVIOR IS DESCRIBED BY A FRACTIONAL-LINEAR LAW

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An approximate solution of the problem of determining the fields of stresses and strain rates due to creep near the tip of a transverse shear crack in a material whose behavior is described by a fractional-linear law of the theory of steady-state creep is given. It is shown that the strain rates have a singularity of the type $\dot{\varepsilon} \sim r^{-\alpha}$ *near the crack tip; the order of singularity* α *changes discretely, depending on the polar angle, and takes the values 1,* $2/3$*, and* $1/2$*.*

Key words: *transverse shear crack, fractional-linear law of steady-state creep, stress field near the crack tip, asymptotics of strain rates.*

Introduction. The analysis of stress and strain (strain-rate) fields in the vicinity of the tip of a crack or an angular cut is one of the main problems in crack mechanics and fracture mechanics. The modern mechanics of deformable solids follows the way of gradual complication and refinement of the constitutive equations that relate the stresses and strains (strain rates) owing to gradual accumulation of experimental data. The power law of creep is widely used in the theory of steady-state creep. Ludwick's exponential law and Nadai's law of a hyperbolic sine function are also known [1]. Principally different constitutive equations of the theory of steady-state creep, where the material constants have a clear physical meaning, were proposed in [2, 3]. A particular case of the constitutive equations proposed is the fractional-linear law of creep, which has the following form in the case of uniaxial loading: $\dot{\varepsilon} = B\sigma/(\sigma_b - \sigma)$, where $\dot{\varepsilon}$ is the creep strain rate, σ is the stress, and B and σ_b are material constants. The constant σ_b can be treated as a certain generalized characteristic of long-time strength on a certain time interval of operation of the structural element considered. The quantity σ_b can also be considered as a characteristic of instantaneous strength determined in experiments with uniaxial fracture. In any case, the value of σ_b is determined rather reliably and has a clear physical meaning. The parameter B is a characteristic determining the strain rate. In processing experimental data, the parameters σ_b and B are determined with a sufficient degree of reliability. The fractional-linear model is physically more grounded than the power-law model proposed by Baily and Norton, and it has some advantages. The fractional-linear model [2, 3] takes into account the maximum critical stress characterizing the instantaneous fracture of the metal at the test temperature and various characteristics of long-time strength under tension and compression. This model also describes the linear creep under low stresses. Moreover, the fractional-linear model takes into account the fact of existence of a nonzero limit of creep, which is a lower restriction for the range of stresses where the creep process is developed. The fractional-linear model of steady-state creep and long-time strength is the subject of numerous investigations [4–6].

From the viewpoint of fracture mechanics, it seems of interest to determine the fields of stresses and strain rates of the creep in the vicinity of the tips of cracks of different types with the use of a fractional-linear approximation. The stress-strain state near the crack tip of anti-plane shear and tensile loading in a material whose behavior is described by a fractional-linear law of creep was considered in [7, 8]. The analysis showed that the strain rates near

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the crack tip are singular, and the order of singularity changes discretely with approaching the crack tip, depending on the polar angle.

1. Formulation of the Problem. Plane Stress State. The objective of the present activities was to study the fields of stresses and strain rates due to creep in the vicinity of the tip of a transverse shear crack in a material whose behavior is described by a fractional-linear law of the theory of steady-state creep under conditions of a plane stress state:

$$\dot\varepsilon_{ij} = \frac{3}{2}\,\frac{f(\sigma_e)}{\sigma_e}\,s_{ij}, \tag{1.1}$$

where $f(\sigma_e) = B\sigma_e/(\sigma_b - \sigma_e)$, B and σ_b are material constants, $\dot\varepsilon_{ij}$ are the components of the tensor of strain rates due to creep, s_{ij} is the stress deviator tensor, and $\sigma_e^2 = 3s_{ij}s_{ij}/2$ is the equivalent stress. In the case of a plane stress state, the constitutive equations of the problem in a polar coordinate system with the pole at the crack tip acquire the form

$$\dot\varepsilon_{rr} = \frac{1}{2}\,B\,\frac{2\sigma_{rr} - \sigma_{\theta\theta}}{\sigma_b - \sigma_e}, \qquad \dot\varepsilon_{\theta\theta} = \frac{1}{2}\,B\,\frac{2\sigma_{\theta\theta} - \sigma_{rr}}{\sigma_b - \sigma_e}, \qquad \dot\varepsilon_{r\theta} = \frac{3}{2}\,B\,\frac{\sigma_{r\theta}}{\sigma_b - \sigma_e}, \tag{1.2}$$

where $\sigma_e^2 = \sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2$.

The equations of equilibrium and the condition of compatibility are presented as

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\,\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \qquad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\,\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\,\frac{\sigma_{r\theta}}{r} = 0; \tag{1.3}$$

$$2\,\frac{\partial}{\partial r}\Big(r\,\frac{\partial\dot\varepsilon_{r\theta}}{\partial\theta}\Big) = \frac{\partial^2\dot\varepsilon_{rr}}{\partial\theta^2} - r\,\frac{\partial\dot\varepsilon_{rr}}{\partial r} + r\,\frac{\partial^2(r\dot\varepsilon_{\theta\theta})}{\partial r^2}. \tag{1.4}$$

The boundary conditions of the problem are the traction-free condition on the crack edges: $\sigma_{\theta\theta}(r, \pm\pi) = 0$ and $\sigma_{r\theta}(r, \pm\pi) = 0$.

By virtue of the problem symmetry, we can study one half-plane and replace the boundary conditions on one crack edge by conditions of symmetry at $\theta = 0$: $\sigma_{rr}(r, 0) = 0$ and $\sigma_{\theta\theta}(r, 0) = 0$.

The stresses become much less intense with distance from the crack tip; therefore, the remote boundary conditions are the conditions of asymptotical approaching the solution of a similar problem for a linearly viscous material:

$$\sigma_{rr} = \sqrt{\frac{C^*\sigma_b}{B2\pi r}}\Big(-5\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\Big), \qquad \sigma_{\theta\theta} = \sqrt{\frac{C^*\sigma_b}{B2\pi r}}\Big(-3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2}\Big),$$

$$\sigma_{r\theta} = \sqrt{\frac{C^*\sigma_b}{B2\pi r}}\Big(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\Big)$$

(C^* is the invariant integral of nonlinear fracture mechanics). To determine the amplitude of the stress field at infinity, we used the fact that the integral C^* is independent of the integration contour. Introducing the dimensionless quantities $\hat\sigma_{ij} = \sigma_{ij}/\sigma_b$, $\hat{\dot\varepsilon}_{ij} = 2\dot\varepsilon_{ij}/B$, $\hat r = r/L$, and $L = C^*/(B\sigma_b)$, we can present the constitutive equations of problem (1.1) in the form

$$\hat{\dot\varepsilon}_{rr} = \frac{2\hat\sigma_{rr} - \sigma_{\theta\theta}}{1 - \hat\sigma_e}, \qquad \hat{\dot\varepsilon}_{\theta\theta} = \frac{2\hat\sigma_{\theta\theta} - \hat\sigma_{rr}}{1 - \hat\sigma_e}, \qquad \hat{\dot\varepsilon}_{r\theta} = \frac{3\hat\sigma_{r\theta}}{1 - \hat\sigma_e}. \tag{1.5}$$

In the dimensionless variables, the remote boundary conditions acquire the following form:

$$\hat\sigma_{rr} = \frac{1}{\sqrt{2\pi\hat r}}\Big(-5\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\Big), \qquad \hat\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi\hat r}}\Big(-3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2}\Big),$$

$$\hat\sigma_{r\theta} = \frac{1}{\sqrt{2\pi\hat r}}\Big(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\Big), \qquad \hat r \to \infty. \tag{1.6}$$

The equations of equilibrium and the compatibility condition remain unchanged. In what follows, the hat sign is omitted.

2. Asymptotic Analysis. In the constitutive equations (1.1), the quantity σ_b is a stress of the type of the strength limit (the components of the stress tensor cannot exceed this value). By analogy with the reasoning in studying problems on anti-plane shear cracks [7] and tensile loading [8], we can assume that the condition characterizing the moment when the critical state is reached $\sigma_e = \sigma_b$ (or $\sigma_e = 1$ in the dimensionless variables) is satisfied only at one point: crack tip. Therefore, the asymptotic expansion of the stress tensor components near the crack mouth ($r \to 0$) can be sought in the form

$$\sigma_{ij}(r,\theta) = \sigma_{ij}^{(0)}(\theta) + r^\alpha \sigma_{ij}^{(1)}(\theta) + o(r^\alpha), \qquad \sigma_e(r,\theta) = 1 - r^\alpha \sigma^{(1)}(\theta) + o(r^\alpha), \tag{2.1}$$

where $\alpha > 0$; $\sigma_{ij}^{(k)}(\theta)$ are functions to be determined. Then, the principal term of the asymptotic expansion of the components of the stress tensor $\sigma_{ij}^{(0)}(\theta)$ is sought from the system of equilibrium equations and the condition of the emergence of the critical state $\sigma_e = 1$. It should be noted that a similar system consisting of two differential equations and one algebraic equation is obtained in studying the stress field in the vicinity of the crack tip in a perfectly plastic material. Substituting (2.1) into the equilibrium equations (1.3) and the condition of the critical state, we obtain the system of ordinary differential equations (ODE)

$$\frac{d\sigma_{r\theta}^{(0)}}{d\theta} + \sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)} = 0, \qquad \frac{d\sigma_{\theta\theta}^{(0)}}{d\theta} + 2\sigma_{r\theta}^{(0)} = 0$$

and the algebraic condition

$$(\sigma_{rr}^{(0)})^2 - \sigma_{rr}^{(0)}\sigma_{\theta\theta}^{(0)} + (\sigma_{\theta\theta}^{(0)})^2 + 3(\sigma_{r\theta}^{(0)})^2 = 1. \tag{2.2}$$

The condition of the critical state (2.2) is satisfied if we assume that

$$\sigma_{rr}^{(0)} = \cos(\theta + c_1)/\sqrt{3}, \quad \sigma_{\theta\theta}^{(0)} = 2\cos(\theta + c_1)/\sqrt{3}, \quad \sigma_{r\theta}^{(0)} = \sin(\theta + c_1)/\sqrt{3}, \tag{2.3}$$

where c_1 is an arbitrary constant. There is also another presentation of the stress tensor components [9, 10] satisfying condition (2.2):

$$\begin{gathered} \sigma_{rr}^{(0)} = a + b\cos 2\vartheta(\theta) + c\sin 2\vartheta(\theta), \qquad \sigma_{\theta\theta}^{(0)} = a - b\cos 2\vartheta(\theta) - c\sin 2\vartheta(\theta), \\ \sigma_{r\theta}^{(0)} = -b\sin 2\vartheta(\theta) + c\cos 2\vartheta(\theta). \end{gathered} \tag{2.4}$$

Here a, b, and c are constants for which the following equality is valid:

$$a^2 + 3b^2 + 3c^2 = 1. \tag{2.5}$$

It should be noted that it is impossible to find a solution of the problem satisfying the boundary conditions on the crack edge and the conditions of symmetry on the crack continuation by using only one of the presentations (2.3) or (2.4). Therefore, we assume that the stress field is described by formulas (2.3) or (2.4) in different characteristic regions of the half-plane $0 \leqslant \theta \leqslant \pi$. The boundaries of these regions are determined from the conditions of continuity of the stress tensor components $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ in passing through these boundaries. The component σ_{rr} can experience a discontinuity. It is found that there exist three characteristic wedge-shaped regions with the boundaries $\theta = \theta_{0a}$ and $\theta = \theta_{0b}$. The stress field is determined by formulas (2.3) in the region $0 \leqslant \theta \leqslant \theta_{0a}$ and by formulas (2.4) in the regions $\theta_{0a} \leqslant \theta \leqslant \theta_{0b}$ and $\theta_{0b} \leqslant \theta \leqslant \pi$. In the sector $\theta_{0b} \leqslant \theta \leqslant \pi$ adjacent to the load-free edge of the crack, the stresses are determined from the traction-free conditions on the upper edge of the crack; hence, the constants a, b, and c are known. The conditions of continuity of the stress tensor components $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ on the rays $\theta = \theta_{0a}$ and $\theta = \theta_{0b}$ are satisfied by choosing appropriate constants a, b, and c of the stress field in the central sector $\theta_{0a} \leqslant \theta \leqslant \theta_{0b}$ and the values of θ_{0a} and θ_{0b}. Indeed, five unknown constants a, b, c, θ_{0a}, and θ_{0b} are subjected to five conditions: requirement (2.5) and four conditions that follow from the continuity of the stress tensor components $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ in passing through the boundaries $\theta = \theta_{0a}$ and $\theta = \theta_{0b}$ of the regions considered. Satisfying two conditions of continuity of the tensor components at $\theta = \theta_{0b}$ owing to an appropriate choice of a, b, and c in the solution for stresses in the sector $\theta_{0a} \leqslant \theta \leqslant \theta_{0b}$ with allowance for Eq. (2.5), the angles θ_{0a} and θ_{0b} can be determined from the conditions of continuity at $\theta = \theta_{0a}$:

$$\begin{gathered} -(8/\sqrt{3})\sin\theta_{0a} = -1 + 3\cos 2\theta_{0b} - (1 + \cos 2\theta_{0b})\cos\theta_{ab} - 2\sin 2\theta_{0b}\sin\theta_{ab}, \\ (4/\sqrt{3})\cos\theta_{0a} = -(1 + \cos 2\theta_{0b})\sin\theta_{ab} + 2\sin 2\theta_{0b}\cos\theta_{ab}, \qquad \theta_{ab} = 2(\theta_{0a} - \theta_{0b}). \end{gathered} \tag{2.6}$$

The solution of system (2.6) was sought numerically by consecutive checking of points of a chosen mesh in a square with the side equal to π, in the plane $(\theta_{0a}, \theta_{0b})$. In this case, this procedure is more preferable than the

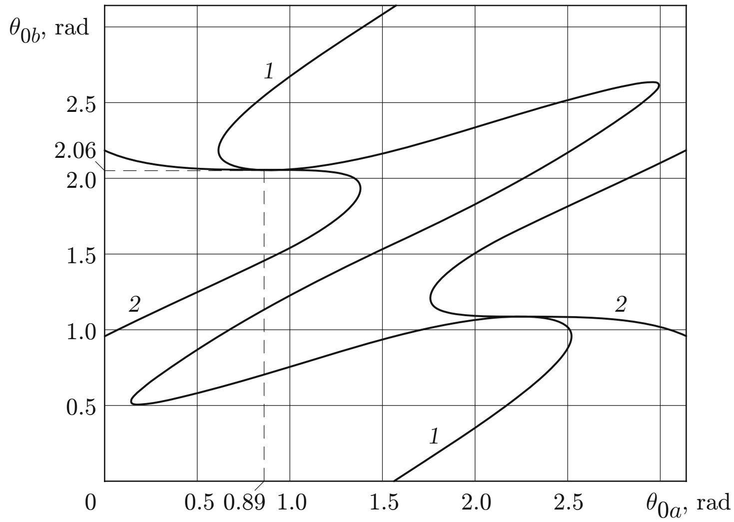

Fig. 1. Graphical solution of the system of algebraic equations (2.6): the curve determined by the first equation in system (2.6) is indicated by 1; the branches of the curve determined by the second equation in system (2.6) is indicated by 2.

method of fastest descent: first, the domain for seeking the roots of the system is specified; second, there is no need to choose the zero approximation, which is important when gradient methods are used. The solution of the system of trigonometric equations (2.6) is the values $\theta_{0a} = 51.06^\circ$ and $\theta_{0b} = 117.80^\circ$. The graphical method of the solution is illustrated in Fig. 1. Under conditions of a plane stress state in a material whose behavior is described by a fractional-linear law of the theory of steady-state creep, the stress distribution in each of the three wedge-shaped regions near the tip of the transverse shear crack is described by the formulas

$$0 \leqslant \theta \leqslant \theta_{0a}\colon \quad \sigma_{rr}^{(0)} = -\frac{1}{\sqrt{3}}\sin\theta, \quad \sigma_{\theta\theta}^{(0)} = -\frac{2}{\sqrt{3}}\sin\theta, \quad \sigma_{r\theta}^{(0)} = \frac{1}{\sqrt{3}}\cos\theta,$$

$$\theta_{0a} \leqslant \theta \leqslant \theta_{0b}\colon \quad \sigma_{rr}^{(0)} = \frac{1}{4}(-1 + 3\cos 2\theta_{0b}) + \frac{1}{4}(1 + \cos 2\theta_{0b})\cos\theta_b + \frac{1}{2}\sin 2\theta_{0b}\sin\theta_b,$$

$$\sigma_{\theta\theta}^{(0)} = \frac{1}{4}(-1 + 3\cos 2\theta_{0b}) - \frac{1}{4}(1 + \cos 2\theta_{0b})\cos\theta_b - \frac{1}{2}\sin 2\theta_{0b}\sin\theta_b, \tag{2.7}$$

$$\sigma_{r\theta}^{(0)} = -\frac{1}{4}(1 + \cos 2\theta_{0b})\sin\theta_b + \frac{1}{2}\sin 2\theta_{0b}\cos\theta_b, \quad \theta_b = 2(\theta - \theta_{0b}),$$

$$\theta_{0b} \leqslant \theta \leqslant \pi\colon \quad \sigma_{rr}^{(0)} = -\frac{1}{2}(1 + \cos 2\theta), \quad \sigma_{\theta\theta}^{(0)} = -\frac{1}{2}(1 - \cos 2\theta), \quad \sigma_{r\theta}^{(0)} = \frac{1}{2}\sin 2\theta.$$

The resultant stress distribution is plotted in Fig. 2. At $\theta = \theta_{0b}$, the stresses σ_{rr} experience a discontinuity. The presented solution can be compared with the stress field in the vicinity of the crack tip for a material whose behavior is described by a power law relating the strains and stresses: $\dot{\varepsilon}_{ij} = 3B\sigma_e^{n-1}s_{ij}/2$ (B and n are material constants), in the limiting case, as $n \to \infty$, which corresponds to a perfectly plastic material.

In accordance with the approach implemented in [11, 12], the solution is sought in the form $\sigma_{ij}(r,\theta) = Kr^{-1/(n+1)}f_{ij}(\theta)$, and the distributions of the stress tensor components over the angle in the vicinity of the crack tip are found by using an ODE system (or one nonlinear ODE if the Airy stress function is used). Directing n to infinity (a numerical simulation normally implies that $n = 100$, 300, or 500), the distributions of the stress tensor components over the angle can be obtained by another method (Fig. 3). A comparison of the distributions of the stress tensor components over the angle shows that the use of both methods of stress field construction yields the same result, which confirms the validity of solution (2.7). Using the stress distribution (2.7) and the constitutive equations (1.5), we can obtain the principal term of the asymptotic expansion of the creep strain rates in the vicinity of the crack tip:

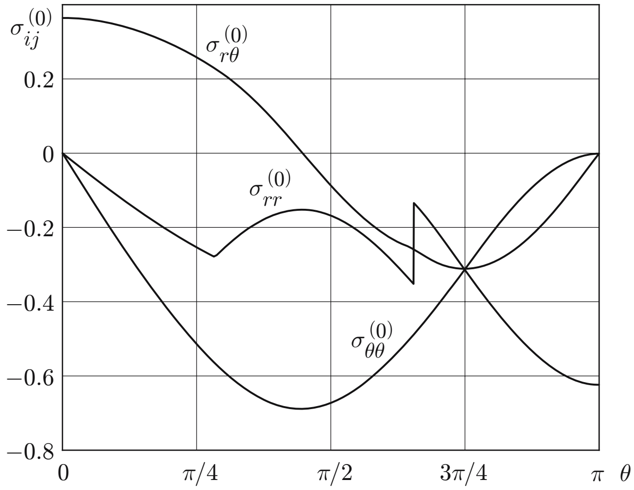

Fig. 2

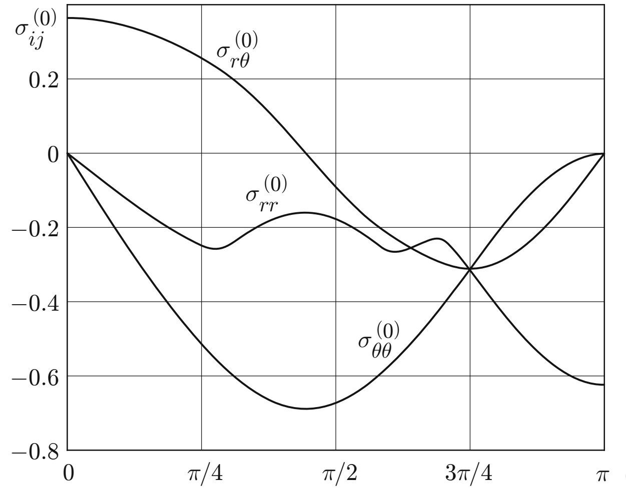

Fig. 3

Fig. 2. Stress tensor components versus the polar angle for a plane stress state (analytical solution of a statically determinable problem).

Fig. 3. Stress tensor components versus the polar angle for a plane stress state [numerical solution of the problem for a material hardened by the power law ($n = 500$)].

$$0 \leqslant \theta < \theta_{0a}: \quad \dot\varepsilon_{rr} = 0, \quad \dot\varepsilon_{\theta\theta} = -\sqrt{3}\,\frac{\sin\theta}{r^\alpha \sigma_1(\theta)}, \quad \dot\varepsilon_{r\theta} = \sqrt{3}\,\frac{\cos\theta}{r^\alpha\sigma_1(\theta)}; \tag{2.8}$$

$$\begin{gathered}\theta_{0a} < \theta < \theta_{0b}: \quad \dot\varepsilon_{rr} = \frac{a + 3b\cos\theta_b + 3c\sin\theta_b}{r^\alpha\sigma_2(\theta)}, \quad \dot\varepsilon_{\theta\theta} = \frac{a - 3b\cos\theta_b - 3c\sin\theta_b}{r^\alpha\sigma_2(\theta)},\\ \dot\varepsilon_{r\theta} = 3\,\frac{-b\sin\theta_b + c\cos\theta_b}{r^\alpha\sigma_2(\theta)},\end{gathered} \tag{2.9}$$

$$a = (-1 + 3\cos 2\theta_{0b})/4, \quad b = (1 + \cos 2\theta_{0b})/4, \quad c = (1/2)\sin 2\theta_{0b};$$

$$\begin{gathered}\theta_{0b} < \theta \leqslant \pi: \quad \dot\varepsilon_{rr} = -\frac{1}{2}\,\frac{1 + 3\cos 2\theta}{r^\alpha\sigma_3(\theta)}, \quad \dot\varepsilon_{\theta\theta} = -\frac{1}{2}\,\frac{1 - 3\cos 2\theta}{r^\alpha\sigma_3(\theta)},\\ \dot\varepsilon_{r\theta} = \frac{3}{2}\,\frac{\sin 2\theta}{r^\alpha\sigma_3(\theta)}.\end{gathered} \tag{2.10}$$

The functions $\sigma_i(\theta)$ ($i = 1, 2, 3$) are found from the compatibility condition (1.4). We can demonstrate by direct calculations that the compatibility condition for $0 \leqslant \theta < \theta_{0a}$ is satisfied for an arbitrary function $\sigma_1(\theta)$ if $\alpha = 1$. If the fractional-linear approximation is used, the stresses in the vicinity of the crack tip are bounded quantities; hence, by virtue of the invariance of the integral C^*, the order of singularity of the creep strain rate should be equal to unity at least in one of the three regions considered.

According to Eqs. (2.8)–(2.10), the creep strain rates are singular near the crack tip ($\alpha > 0$). At this stage of the solution, the order of singularity of strain rates is yet unknown, and the following reasoning can be used for determining this order.

Similar to the analysis in studying the problems on cracks of anti-plane shear and tensile loading, the asymptotic solution of the problem can be sought on the basis of asymptotic expansions of the displacement rates $u_r(r,\theta)$ and $u_\theta(r,\theta)$ in the vicinity of the crack tip ($r \to 0$):

$$u_r(r,\theta) = f_0(\theta) + r^\alpha f_1(\theta) + \ldots, \qquad u_\theta(r,\theta) = g_0(\theta) + r^\alpha g_1(\theta) + \ldots \qquad (\alpha > 0). \tag{2.11}$$

The asymptotic expansions of the displacement rates of the form (2.11) allow us to describe the stress field bounded near the crack tip and the singular field of the creep strain rates:

$$\dot\varepsilon_{rr}(r,\theta) = r^{\alpha-1}\varepsilon_{rr}^{(1)}(\theta) + \ldots, \qquad \dot\varepsilon_{\theta\theta}(r,\theta) = r^{-1}\varepsilon_{\theta\theta}^{(0)}(\theta) + r^{\alpha-1}\varepsilon_{\theta\theta}^{(1)}(\theta) + \ldots,$$
$$\dot\varepsilon_{r\theta}(r,\theta) = r^{-1}\varepsilon_{r\theta}^{(0)}(\theta) + r^{\alpha-1}\varepsilon_{r\theta}^{(1)}(\theta) + \ldots, \tag{2.12}$$

where

$$\varepsilon_{rr}^{(1)}(\theta) = \alpha f_1(\theta), \qquad \varepsilon_{\theta\theta}^{(0)}(\theta) = f_0(\theta) + g_0'(\theta), \qquad \varepsilon_{\theta\theta}^{(1)}(\theta) = f_1(\theta) + g_1'(\theta),$$
$$\varepsilon_{r\theta}^{(0)}(\theta) = [f_0'(\theta) - g_0(\theta)]/2, \qquad \varepsilon_{r\theta}^{(1)}(\theta) = [f_1'(\theta) + (\alpha-1)g_1(\theta)]/2.$$

The functions u_r and u_θ to be determined are found from the equilibrium equations where the stress tensor components are expressed via the components of the creep strain rate tensor with the use of the constitutive equations of problem (1.5). Substituting the asymptotic expansions (2.12) into the resultant equations and identifying the coefficients at the minimum power of r, we obtain the following system of two ODEs:

$$\varepsilon_{\theta\theta}^{(0)}[\varepsilon_{rr}^{(0)}(\varepsilon_{r\theta}^{(0)})' - \varepsilon_{r\theta}^{(0)}(\varepsilon_{\theta\theta}^{(0)})' - (\varepsilon_{\theta\theta}^{(0)})^2 - (\varepsilon_{r\theta}^{(0)})^2] = 0,$$
$$\varepsilon_{r\theta}^{(0)}[\varepsilon_{rr}^{(0)}(\varepsilon_{r\theta}^{(0)})' - \varepsilon_{r\theta}^{(0)}(\varepsilon_{\theta\theta}^{(0)})' - (\varepsilon_{\theta\theta}^{(0)})^2 - (\varepsilon_{r\theta}^{(0)})^2] = 0. \tag{2.13}$$

The equations of system are valid in the following cases: 1) $\varepsilon_{\theta\theta}^{(0)} \neq 0$, $\varepsilon_{r\theta}^{(0)} \neq 0$ and $\varepsilon_{rr}^{(0)}(\varepsilon_{r\theta}^{(0)})' - \varepsilon_{r\theta}^{(0)}(\varepsilon_{\theta\theta}^{(0)})' - (\varepsilon_{\theta\theta}^{(0)})^2 - (\varepsilon_{r\theta}^{(0)})^2 = 0$; 2) $\varepsilon_{\theta\theta}^{(0)} = 0$ and $\varepsilon_{r\theta}^{(0)} = 0$.

In the first case, the strain rates have a singularity of the form r^{-1}, and the stress tensor components tend to finite values as $r \to 0$:

$$\sigma_{\theta\theta} \to \frac{1}{\sqrt3}\frac{2\varepsilon_{\theta\theta}^{(0)}}{\varepsilon}, \qquad \sigma_{rr} \to \frac{1}{\sqrt3}\frac{\varepsilon_{\theta\theta}^{(0)}}{\varepsilon}, \qquad \sigma_{r\theta} \to \frac{1}{\sqrt3}\frac{\varepsilon_{r\theta}^{(0)}}{\varepsilon},$$

where $\varepsilon^2 = (\varepsilon_{\theta\theta}^{(0)})^2 + (\varepsilon_{r\theta}^{(0)})^2$. The unknown functions $\varepsilon_{\theta\theta}^{(0)}$ and $\varepsilon_{r\theta}^{(0)}$ are determined from the ODEs that follow from the equations of equilibrium of the problem $(\varepsilon_{r\theta}^{(0)}/\varepsilon)' - \varepsilon_{\theta\theta}^{(0)}/\varepsilon = 0$, $(\varepsilon_{\theta\theta}^{(0)}/\varepsilon)' + \varepsilon_{r\theta}^{(0)}/\varepsilon = 0$, whose solutions are the functions

$$\varepsilon_{\theta\theta}^{(0)}(\theta) = \rho(\theta)(-A_1\sin\theta + A_2\cos\theta), \qquad \varepsilon_{r\theta}^{(0)}(\theta) = \rho(\theta)(A_1\cos\theta + A_2\sin\theta)$$

[$\rho(\theta)$ is an arbitrary function and A_1 and A_2 are constants of integration]. Thus, in the case considered, the stresses and strain rates in the vicinity of the crack tip are presented as

$$\sigma_{\theta\theta} = 2(-A_1\sin\theta + A_2\cos\theta)/\sqrt3, \qquad \sigma_{rr} = (-A_1\sin\theta + A_2\cos\theta)/\sqrt3,$$
$$\sigma_{r\theta} = (A_1\cos\theta + A_2\sin\theta)/\sqrt3, \tag{2.14}$$
$$\dot\varepsilon_{rr} = 0, \quad \dot\varepsilon_{\theta\theta} = \frac{\rho(\theta)}{r}(-A_1\sin\theta + A_2\cos\theta), \quad \dot\varepsilon_{r\theta} = \frac{\rho(\theta)}{r}(A_1\cos\theta + A_2\sin\theta).$$

In the second case, where $\varepsilon_{\theta\theta}^{(0)} = 0$ and $\varepsilon_{r\theta}^{(0)} = 0$, the expressions for the creep strain rates in the vicinity of the crack tip have the form

$$\dot\varepsilon_{rr}(r,\theta) = r^{\alpha-1}\varepsilon_{rr}^{(1)} + \ldots, \quad \dot\varepsilon_{\theta\theta}(r,\theta) = r^{\alpha-1}\varepsilon_{\theta\theta}^{(1)} + \ldots, \quad \dot\varepsilon_{r\theta}(r,\theta) = r^{\alpha-1}\varepsilon_{r\theta}^{(1)} + \ldots\ . \tag{2.15}$$

As $r \to 0$, the stress tensor components are determined by the formulas

$$\sigma_{\theta\theta} = (\varepsilon_{rr}^{(1)} + 2\varepsilon_{\theta\theta}^{(1)})/(\sqrt3\,\varepsilon^{(1)}), \quad \sigma_{rr} = (\varepsilon_{\theta\theta}^{(1)} + 2\varepsilon_{rr}^{(1)})/(\sqrt3\,\varepsilon^{(1)}), \quad \sigma_{r\theta} = \varepsilon_{r\theta}^{(1)}/(\sqrt3\,\varepsilon^{(1)}),$$

where $(\varepsilon^{(1)})^2 = (\varepsilon_{\theta\theta}^{(1)})^2 + (\varepsilon_{rr}^{(1)})^2 + \varepsilon_{\theta\theta}^{(1)}\varepsilon_{rr}^{(1)} + (\varepsilon_{r\theta}^{(1)})^2$.

The system of equations for determining the functions $\varepsilon_{ij}^{(1)}$ is composed from equations that follow from the equilibrium equations and compatibility condition for strains and are obtained with the use of expansions (2.15). Performing appropriate calculations, we can obtain expressions of the form (2.4) for the stress tensor components. Combining presentations (2.4) and (2.14) for the stresses and providing satisfaction of the boundary conditions on the crack edges, conditions of symmetry on its continuation, and requirements of continuity of the components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ on the rays separating the sectors, we can find the stress distribution (2.7). The following presentations are valid for the strain rates in the vicinity of the crack tip:

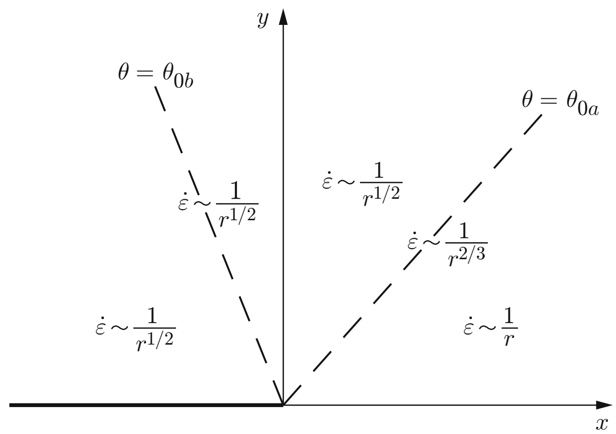

Fig. 4. Discrete change in the order of singularity of the creep strain rates versus the polar angle in the vicinity of the crack tip for a plane stress state.

$$
\begin{aligned}
0 \leqslant \theta < \theta_{0a}: \quad & \dot\varepsilon_{rr} = 0, \quad \dot\varepsilon_{\theta\theta} = -\frac{\rho_1(\theta)\sin\theta}{r}, \quad \dot\varepsilon_{r\theta} = \frac{\rho_1(\theta)\cos\theta}{r}, \\
\theta_{0a} < \theta < \theta_{0b}: \quad & \dot\varepsilon_{rr} = \rho_2(\theta)\,\frac{a + 3b\cos\theta_b + 3c\sin\theta_b}{r^{1-\alpha}}, \quad \dot\varepsilon_{\theta\theta} = \rho_2(\theta)\,\frac{a - 3b\cos\theta_b - 3c\sin\theta_b}{r^{1-\alpha}}, \\
& \dot\varepsilon_{r\theta} = 3\rho_2(\theta)\,\frac{-b\sin\theta_b + c\cos\theta_b}{r^{1-\alpha}}, \\
\theta_{0b} < \theta \leqslant \pi: \quad & \dot\varepsilon_{rr} = -\rho_3(\theta)\,\frac{1 + 3\cos 2\theta}{r^{1-\alpha}}, \quad \dot\varepsilon_{\theta\theta} = -\rho_3(\theta)\,\frac{1 - 3\cos 2\theta}{r^{1-\alpha}}, \\
& \dot\varepsilon_{r\theta} = 3\rho_3(\theta)\frac{\sin 2\theta}{r^{1-\alpha}}.
\end{aligned} \tag{2.16}
$$

The asymptotics of the strain rates (2.8)–(2.10) and (2.16) are solutions of the same problem; hence, they have to coincide. A comparison of Eqs. (2.8)–(2.10) and (2.16) shows that the strain rates have a singularity in the vicinity of the crack tip, and the order of singularity changes discretely as a function of the polar angle θ and acquires the value $\alpha = 1$ for $0 \leqslant \theta < \theta_{0a}$; in the remaining sectors ($\theta_{0a} < \theta < \theta_{0b}$ and $\theta_{0b} < \theta \leqslant \pi$), the order of singularity acquires the value $\alpha = 1/2$ (Fig. 4).

In the interval $\theta_{0a} < \theta < \theta_{0b}$, the function $\sigma_2(\theta)$ is determined as the solution of the linear differential equation with variable coefficients

$$
\begin{aligned}
&(a + 3b\cos\theta_b + 3c\sin\theta_b)y_2'' + 6(\alpha+1)(-b\sin\theta_b + c\cos\theta_b)y_2' \\
&\qquad + \alpha[\alpha a - 3(\alpha+2)(b\cos\theta_b + c\sin\theta_b)]y_2 = 0,
\end{aligned} \tag{2.17}
$$

where $y_2 = 1/\sigma_2$. The solution of Eq. (2.17) can be presented in the form

$$
\begin{aligned}
y_2(\theta) = \Big[C_1\Big(\frac{d + 3c - \tan\theta_b}{d - 3c + \tan\theta_b}\Big)^{1/4} + C_2\Big(\frac{d + 3c - \tan\theta_b}{d - 3c + \tan\theta_b}\Big)^{-1/4}\Big](1 + \tan^2\theta_b)^{1/4} \\
\times (a + a\tan^2\theta_b + 3b - 3b\tan^2\theta_b + 6c\tan\theta_b)^{-1/4},
\end{aligned}
$$

where $d = \sqrt{9b^2 + 9c^2 - a^2}$. Inspection of the latter expression shows that the asymptotic expansion of the strain rates (2.9) is not uniformly suitable for $\theta = \theta_{0a}$ because the function $a + a\tan^2\theta_b + 3b - 3b\tan^2\theta_b + 6c\tan\theta_b$ contained in the asymptotics of the strain rates vanishes at this value of the polar angle. Thus, the asymptotic expansion (2.9) is uniformly suitable everywhere on the interval $\theta_{0a} < \theta \leqslant \theta_{0b}$, except for a small neighborhood of the ray $\theta = \theta_{0a}$. To determine the asymptotics of the strain rates on the ray $\theta = \theta_{0a}$, we have to assume that $\theta = \theta_{0a} + r^\mu a$ and consider the neighborhood of this ray under the condition of continuity of the components of the strain rate tensor $\dot\varepsilon_{\theta\theta}$ and $\dot\varepsilon_{r\theta}$. Formulating the conditions of continuity of the component $\dot\varepsilon_{r\theta}$ on the ray $\theta = \theta_{0a}$ and considering the neighborhood of this ray, we can find that $\mu = 1/3$ and, hence, the creep strain rates on the

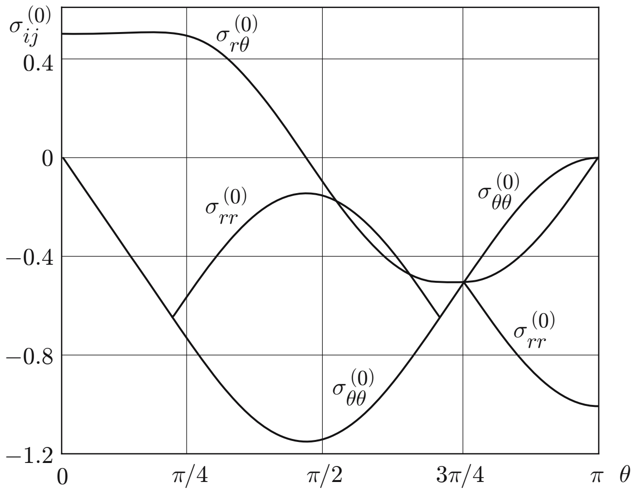

Fig. 5. Components of the strain tensor versus the polar angle in the case of plane strains (analytical solution of the statically determinable problem).

ray $\theta = \theta_{0a}$ are proportional to $r^{-2/3}$. At $\theta_{0b} < \theta \leqslant \pi$, the condition of compatibility allows us to obtain the ODEs with respect to the function $\sigma_3(\theta)$:

$$(1 + 3\cos 2\theta)y_3'' - 6(\alpha + 1)\sin 2\theta y_3' + \alpha[\alpha - 3(\alpha + 2)\cos 2\theta]y_3 = 0. \tag{2.18}$$

Here $y_3 = y_3(\theta) = 1/\sigma_3(\theta)$. The solution of Eq. (2.18) is the function

$$y_3 = (1 + 3\cos 2\theta)^{-1/(2\alpha)}\Big[C_1\Big(\frac{\sin\theta + \sqrt{2}\,\cos\theta}{-\sin\theta + \sqrt{2}\,\cos\theta}\Big)^{1/(2\alpha)} + C_2\Big(\frac{-\sin\theta + \sqrt{2}\,\cos\theta}{\sin\theta + \sqrt{2}\,\cos\theta}\Big)^{1/(2\alpha)}\Big],$$

which has no singularities at $\theta_{0b} \leq \theta \leq \pi$.

Formulas (2.7)–(2.10) describe the fields of stresses and strain rates in the vicinity of the crack tip. This near field has to be matched with the external field, which depends on the system of external loads applied to the solid and on of geometry of the real sample. The matching procedure can be performed by using the property of invariance of the integral C^*. In this case, we have to choose two contours including the crack tip; one of the contours has to be located in the vicinity of the crack tip, and the value of the integral C^* has to be found with the use of the solution valid in the crack neighborhood (2.4), (2.8)–(2.10); the other contour has to be located at a certain distance from the crack tip, and the integral C^* has to be found on the basis of the far field (1.6). For the problem considered, we obtain

$$2\int_0^{\theta_{0a}} \rho_1(\theta)(1 - \sqrt{3}\,\sin^2\theta)\cos\theta\,d\theta = 1.$$

3. Plane Strain State. Using the above-developed approach for the transverse shear crack under conditions of a plane strain state, we can find the stress field near the crack tip:

$$\begin{aligned}
0 \leqslant \theta \leqslant \theta_{0a}&: \quad \theta_{0a} = \pi/8 + 1/4, \quad \sigma_{\theta\theta}^{(0)} = \sigma_{rr}^{(0)} = -\theta, \quad \sigma_{r\theta}^{(0)} = 1/2,\\
\theta_{0a} \leqslant \theta \leqslant \theta_{0b}&: \quad \theta_{0b} = 5\pi/8 + 1/4, \quad \sigma_{rr}^{(0)} = -1/4 - \pi/8 - (1/2)\cos\Theta,\\
&\quad \sigma_{\theta\theta}^{(0)} = -1/4 - \pi/8 + (1/2)\cos\Theta,\\
&\quad \sigma_{r\theta}^{(0)} = (1/2)\sin\Theta, \quad \Theta(\theta) = 2\theta + \pi/4 - 1/2,\\
\theta_{0b} \leqslant \theta \leqslant 3\pi/4&: \quad \sigma_{\theta\theta}^{(0)} = \sigma_{rr}^{(0)} = \theta - 1/2 - 3\pi/4, \quad \sigma_{r\theta}^{(0)} = -1/2,\\
3\pi/4 \leqslant \theta \leqslant \pi&: \quad \sigma_{rr}^{(0)} = -(1 + \cos 2\theta)/2, \quad \sigma_{\theta\theta}^{(0)} = -(1 - \cos 2\theta)/2,\\
&\quad \sigma_{r\theta}^{(0)} = (1/2)\sin 2\theta.
\end{aligned} \tag{3.1}$$

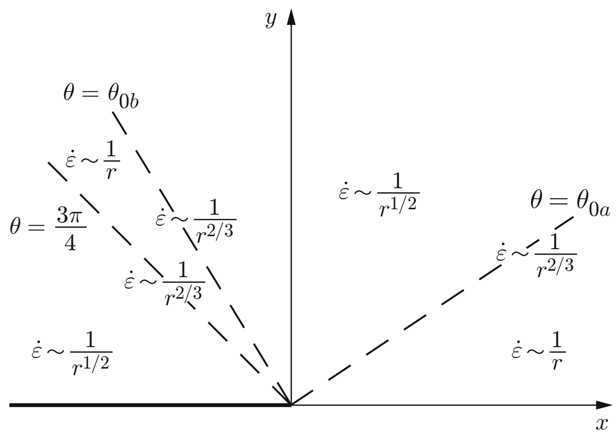

Fig. 6. Discrete change in the order of singularity of the creep strain rates versus the polar angle in the vicinity of the crack tip for the plane strain state.

The stress field (3.1) coincides with the known stress field in the vicinity of the crack tip under conditions of a plane strain state in a perfectly plastic rigid material [11] (Fig. 5). Using the stress distribution obtained (3.1), we can find the components of the strain rate tensor near the crack mouth:

$$0 \leqslant \theta < \theta_{0a}\colon \quad \theta_{0a} = \pi/8 + 1/4, \quad \dot\varepsilon_{rr} = \dot\varepsilon_{\theta\theta} = 0, \quad \dot\varepsilon_{r\theta} = 1/(r^\alpha \sigma_1(\theta)),$$

$$\theta_{0a} < \theta < \theta_{0b}\colon \quad \theta_{0b} = 5\pi/8 + 1/4,$$

$$\dot\varepsilon_{\theta\theta} = \cos\Theta\,[D_1 \cos^{-\alpha}(\Theta/2 + \pi/4) + E_1 \sin^{-\alpha}(\Theta/2 + \pi/4)]/r^\alpha,$$

$$\dot\varepsilon_{r\theta} = \sin\Theta\,[D_1 \cos^{-\alpha}(\Theta/2 + \pi/4) + E_1 \sin^{-\alpha}(\Theta + \pi/4)]/r^\alpha,$$

$$\theta_{0b} < \theta < 3\pi/4\colon \quad \dot\varepsilon_{rr} = \dot\varepsilon_{\theta\theta} = 0, \quad \dot\varepsilon_{r\theta} = 1/(r^\alpha \sigma_3(\theta)),$$

$$3\pi/4 < \theta \leqslant \pi\colon \quad \dot\varepsilon_{\theta\theta} = -\dot\varepsilon_{rr} = \cos 2\theta [D_2 \cos^{-\alpha}(\theta + \pi/4) + E_2 \sin^{-\alpha}(\theta + \pi/4)]/r^\alpha,$$

$$\dot\varepsilon_{r\theta} = \sin(2\theta)[D_2 \cos^{-\alpha}(\theta + \pi/4) + E_2 \sin^{-\alpha}(\theta + \pi/4)]/r^\alpha.$$

As in the case of a plane stress state, $\alpha = 1$ for $0 \leqslant \theta < \theta_{0a}$ and $\theta_{0b} < \theta < 3\pi/4$ and $\alpha = 1/2$ for $\theta_{0a} < \theta < \theta_{0b}$ and $3\pi/4 < \theta \leqslant \pi$. On the interfaces between these regions, $\theta = \theta_{0a}$, $\theta = \theta_{0b}$, and $\theta = 3\pi/4$, the order of singularity of the components of the strain rate tensor equals 2/3. The asymptotic analysis performed shows that the specific feature of the fractional-linear law is a discrete dependence of the strain rate asymptotics on the polar angle as the crack tip is approached (Fig. 6).

4. Conclusions and Discussion. The asymptotic analysis of the fields of stresses and strain rates due to creep near the transverse shear crack tip under conditions of plane stress and plane strain with the use of fractional-linear approximation showed that the stress fields near the crack mouth are defined by different functional dependences for different wedge-shaped regions of variation of the polar angle. In the limiting case (when the order of nonlinearity tends to infinity), the stress fields are demonstrated to coincide with the stress distribution near the crack tip in a material whose behavior is described by a power law of creep, which confirms the solution obtained for a statically determinable problem. The solution obtained allows us to explain the character of the strain field near the crack mouth in a perfectly plastic material, because the condition of reaching the critical state $\sigma_e = \sigma_b$ is similar to the condition of the emergence of the Mises plastic flow. In contrast to problems of the theory of perfect plasticity in determining the kinematics of plastic flow in the vicinity of the crack tip, where it is possible to determine only some singularities of the strain field, the present approach allows the velocity field in each sector to be determined.

The solutions obtained supplement the existing class of solutions of problems on the stress state in the vicinity of the crack tip in a material whose behavior is described by a fractional-linear relation.

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